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Controller assessment for a class of non-linear systems

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Abstract

The use of autoregressive moving average (ARMA) models to assess the control loop performance for processes that are adequately described by the superposition of a linear dynamic model and linear stochastic or deterministic disturbance model is well known. In this paper, classes of non-linear dynamic/stochastic systems for which a similar result can be obtained are established for single-input single-output discrete system. For these systems, lower mean-square error bounds on performance, can be estimated from the closed-loop routine operating data by using non-linear autoregressive moving average with exogenous inputs (NARMAX) models. It is necessary to know the process time delay. The fitting of these models is greatly facilitated by using efficient algorithms, such as Orthogonal Least Squares or other fast orthogonal search algorithms. These models can also be used to assess the predictive importance of non-linearities over multiple-time horizons.

Keywords: Non-linear performance assessment; Non-linear minimum variance control; NARMAX model; Orthogonal Least Squares

1. Introduction

There has been considerable research and industrial application in performance monitoring and assessment in the last decade. While research has focused on monitoring and assessment for univariate and multivariate systems, most industrial applications, and especially those using commercial packages, use a univariate approach [1]. These methods assume that the closed-loop system can be described by a linear difference equation, driven by either stochastic or deterministic disturbances. This type of description arises when the process is adequately described by the superposition of a linear(ized) transfer function model plus additive stochastic or deterministic disturbances. When a linear controller is used, the closed-loop is described by a linear transfer function.

Far less has been written on extending the methodologies for performance assessment to non-linear systems. There are several challenges:

(1) Complexity of non-linear behavior.
Non-linear processes can exhibit six general types of behavior [2,3]:
  (a) Harmonics arising from periodic inputs.
  (b) Subharmonics arising from periodic inputs.
  (c) Chaotic response to simple inputs.
  (d) Input-dependent stability.
  (e) Asymmetric response to symmetric inputs.
  (f) Steady-state input and output multiplicities.
(2) Non-equivalent representations.
It is a standard result [4], that any time-invariant linear system can be completely characterized by its impulse response, or equivalently by an autoregressive model. Unfortunately, this equivalence cannot be extended to all non-linear problems [5].
(3) Disturbance representation.
For any process, disturbances can enter at any point. For linear systems, these disturbances can always be

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represented as an additive output disturbance. Since superposition does not exist in non-linear systems, this representation is not universal. This presents challenges both in modelling and in the determination of the minimum variance performance bound.

(4) Challenges in model determination and parameter estimation.
For systems that admit a linear representation, there are well-established methods for obtaining models of the closed-loop system. These methods can be automated. For non-linear systems there are enormously rich classes of models to be entertained, and the resulting models often have many parameters that must be estimated.

In this paper, some preliminary results are presented for determining the minimum variance performance bound for a class of non-linear systems. The basis for minimum variance performance bounds was developed in [6]. There it was shown that the minimum variance performance bound for a linear system could be estimated from routine closed-loop data by fitting a time series to the outputs and then determining the variance of the b-step ahead predictor, where b is the process delay. The underlying theory relies on the development of minimum variance controllers, outlined in [7,8], and the existence of a feedback invariant [6]. The feedback invariant is a dynamic component of the closed-loop system that is not affected by feedback. In the case of linear systems, the feedback invariant can be easily recovered from a time series description of the closed-loop system. The feedback invariant is then used to estimate the variance of the output that would be achieved if a minimum variance controller were to be implemented.

The development of non-linear minimum variance controllers has been reported in [9] for processes that admit a non-linear ARMAX representation. As will be shown in this paper, the minimum-variance-feedback invariant does not exist for the general process described in [9]. The design of a minimum variance controller for a class of non-linear system described by the superposition of a non-linear dynamic model and a linear, output stochastic disturbance model has been described in [10,11]. Grimble [11] notes that the use of an output representation for a disturbance model is very much motivated by pragmatic reasons.

In this paper, it is shown that a minimum-variance-feedback invariant exists for an important class of non-linear processes that can be described by the superposition of a non-linear dynamic model and additive linear or partially non-linear disturbance. In these instances, the minimum variance performance can be estimated from routine operating data. It may be necessary to fit a time series model to the closed-loop data using both inputs and outputs. In many cases, the parameters of the closed-loop system can be estimated using linear regression techniques. Included in this class of processes are those whose process dynamics can be described by an autoregressive Volterra series [2]. Such models are capable of representing processes that exhibit harmonics, asymmetric behavior – including asymmetric dead zones that are typically encountered with valve stiction, and input multiplicities.

The outline of this paper is as follows: In Section 2, a general non-linear input–output model is introduced. In Section 3, a review of linear minimum variance controllers and the estimation of performance lower bounds from routine operating data is provided. This is followed by the development of non-linear minimum variance controllers and a development that shows the existence of a feedback invariant for this system. In Section 5, a more detailed description of the Volterra representation and one general method are outlined for estimating the minimum variance performance bounds. This is followed by a simulation example to demonstrate the essential features of the method. The paper is concluded with a description of outstanding issues and limitations of the proposed methodology.

2. Process description

The class of models considered in this paper are those that can be modelled using an input/output representation. State models are excluded. For discrete models, the general form of the description is:

$$z_t = f_1(z^*_{t-1}, u^*_{t-b}) \tag{1}$$

where z_t is the deterministic output of the system in response to the inputs that are denoted by u_t. b represents the number of whole periods of delay in the system and is the number of sampling intervals that elapse between making a change in the process input and first observing its effect. The notation $f_1(z^*_{t-1}, u^*_{t-b})$ denotes a function of previous values of z_{t-i}, $i = 1..n_z$, denoted by z^*_{t-1} and u_{t-b-j}, $j = 0..n_u$, denoted by u^*_{t-b}. Common representations for discrete systems arise when $f_1(z^*_{t-1}, u^*_{t-b})$ is expanded in finite polynomials involving summations of terms z_{t-i}, $z_{t-i}z_{t-j}$, $\ldots u_{t-b-i}$, $u_{t-b-i}u_{t-b-j}, \ldots,$ $z_{t-i}u_{t-b-j} \ldots$ [12,13,9]. The resulting expansions produce non-linear difference equations. Hammerstein and Weiner systems are encompassed in this framework [5,9,14].

In this paper we are interested in systems that are also affected by disturbances. The most elementary representation is:

$$y_t = z_t + a_t = f_1(z^*_{t-1}, u^*_{t-b}) + a_t \tag{2}$$

where y_t is the measured output and a_t is a white noise element, sometimes also known as the innovation sequence. As noted by Sales and Billings [9], this representation cannot be estimated from input/output data since the z's are not observed. Instead, the authors note that the appropriate input/output representation is:

$$y_t = f_2(y^*_{t-1}, u^*_{t-b}) + a_t \tag{3}$$

Eq. (3) provides a restricted description for an additive disturbance as it does not allow for a general disturbance that might include moving average terms or cross-products between the stochastic driving force and the inputs and outputs. The most general form is [9,13]:

$$y_t = f_3(y^*_{t-1}, u^*_{t-b}, a^*_{t-1}) + a_t \tag{4}$$

In linear systems, the effect of process disturbances can always be correctly represented as an output disturbance regardless of where they actually appear in the system. This is a consequence of the principle of superposition. In non-linear systems, superposition does not hold. It is useful however to provide an additive description, which is of the form:

$$y_t = f_P(y^*_{t-1}, u^*_{t-b}) + D_t \tag{5}$$

where D_t is the additive distance which can be represented by a non-linear ARMA model as $D_t = f_D(D^*_{t-1}, a^*_{t-1}) + a_t$.

3. Minimum variance performance bounds and feedback invariants for linear systems

3.1. Minimum variance performance bounds

In linear systems, the following representation is often used

$$y_t = \frac{\omega(q^{-1})}{\delta(q^{-1})} u_{t-b} + D_t \tag{6}$$

where $\omega(q^{-1})$ and $\delta(q^{-1})$ are polynomials in the backshift operator q^{-1}, defined such that $q^{-1}y_t = y_{t-1}$. The disturbance D_t is modelled as the output of a linear filter driven by white noise. This type of disturbance is conveniently represented by an autoregressive-integrated-moving-average (ARIMA) time series model of order (n_ϕ, d, n_θ) of the form:

$$D_t = \frac{\theta(q^{-1})}{\phi(q^{-1})\nabla^d} a_t = \psi(q^{-1}) a_t \tag{7}$$

where $\nabla = (1 - q^{-1})$ is the difference operator and d is a non-negative integer. Typically $d \leqslant 2$. Positive values of d allow for drifting and other forms of non-stationary behavior. $\theta(q^{-1})$ and $\phi(q^{-1})$ are monic and stable polynomials.

The design of controllers to minimize the variance of the process output can be traced to [7,8]. To derive the minimum variance controller, Eqs. (6) and (7) are substituted into Eq. (5) to obtain:

$$\begin{aligned} y_{t+b} &= \frac{\omega(q^{-1})}{\delta(q^{-1})} u_t + D_{t+b} = \frac{\omega(q^{-1})}{\delta(q^{-1})} u_t + \widehat{D}_{t+b/t} + e_{t+b/t} \\ &= \hat{y}_{t+b/t} + e_{t+b/t} \end{aligned} \tag{8}$$

$\widehat{D}_{t+b/t}$ and $\hat{y}_{t+b/t}$ are the b-step ahead minimum-mean-square-error forecast for the disturbance and y_{t+b} respectively and $e_{t+b/t}$ is the prediction error. These terms are constructed by the methods described in [7,8]. The prediction error, $e_{t+b/t}$, is a moving average process of order $b-1$.

$$e_{t+b/t} = (1 + \varphi_1 q^{-1} + \cdots + \varphi_{b-1} q^{-(b-1)}) a_{t+b} \tag{9}$$

The φ weights are the first $b-1$ impulse coefficients of the disturbance transfer function in Eq. (7). With these developments, *if it is possible* to choose the control action such that

$$\frac{\omega(q^{-1})}{\delta(q^{-1})} u_t + \widehat{D}_{t+b/t} = 0 \tag{10}$$

then the process output equals the prediction error. The process output under this control scheme will be denoted by y^{MV}_{t+b}.

$$y^{\mathrm{MV}}_{t+b} = e_{t+b/t} \tag{11}$$

The prediction error, $e_{t+k/t}$ does not depend on the manipulated variable over the prediction interval $k = 1..b$. The lower bound on performance, as measured in the mean square sense, is:

$$\sigma^2_{\mathrm{MV}} = \mathrm{var}\{y^{\mathrm{MV}}_{t+b}\} = (1 + \varphi_1^2 + \cdots + \varphi_{b-1}^2)\sigma_a^2 \tag{12}$$

If it is not possible to invert $\frac{\omega(q^{-1})}{\delta(q^{-1})}$, then the output variance exceeds that of the lower bound.

Conditional expectation interpretations. The development of the minimum variance controller and controller bound can be cast in terms of conditional expectations. The process in Eq. (8) has been written as:

$$y_{t+b} = \hat{y}_{t+b/t} + e_{t+b/t} \tag{13}$$

where $\hat{y}_{t+b/t}$ is the prediction of y_{t+b} and is uncorrelated with the prediction error $e_{t+b/t}$. When the $\{a_t\}$ are iid Normal variables, then $\hat{y}_{t+b/t} = E\{y_{t+b}|I_t\}$. $E\{y_{t+b}|I_t\}$ denotes the conditional mean of y_{t+b} using all of the information available up to an including time t. It is well known that the conditional mean has the lowest prediction error variance among all predictors that use the same information set [7,15,16]. In this case, the minimum variance controller can be seen as a strategy to choose the current control action to set the conditional mean to zero, thus minimizing the variance of y_{t+b}. When the driving force is white noise, then the prediction error has the smallest variance among all linear predictors [7,16]. In the latter case, the difficulties in constructing the conditional mean arise from the challenge in constructing the joint distribution of all of random variables and then determining the conditional mean from the joint distribution. Consequently, the formulae used in this section for the prediction and prediction error are most often used.

3.2. Feedback invariants

If the process is controlled by a linear feedback controller, the transfer function between the measured output and the driving force for the disturbance is of the form

$$\tilde{y}_t = \frac{\alpha(q^{-1})}{\beta(q^{-1})} a_t \tag{14}$$

where $\tilde{y}_t = y_t - y_{\mathrm{sp}}$. It was shown in [6] that the closed loop can be written as:

$$\tilde{y}_{t+b} = \frac{\alpha(q^{-1})}{\beta(q^{-1})} a_{t+b} = e_{t+b/t} + \frac{\gamma(q^{-1})}{\beta(q^{-1})} a_t$$
$$= e_{t+b/t} + \frac{\gamma(q^{-1})}{\alpha(q^{-1})} \tilde{y}_t \tag{15}$$

The term $e_{t+b/t}$ appears in the closed-loop. It is not influenced by feedback and hence is termed a feedback invariant. Consequently, this term can be estimated from routine operating data in one of two ways.

(1) Estimation of closed-loop impulse response [6]. In this approach a time series model of the form given in Eq. (14) is fit to closed-loop data. The first $b-1$ impulse coefficients of $\frac{\alpha(q^{-1})}{\beta(q^{-1})}$ are estimates of the first $b-1$ coefficients of the open-loop disturbance transfer function. Using the estimated coefficients and an estimate of σ_a^2 obtained from the model-estimation stage, the minimum variance performance can be estimated [6].

(2) Direct estimation [17]. In this approach a lagged regression of the form:

$$\tilde{y}_{t+b} = e_{t+b/t} + \xi(q^{-1})\tilde{y}_t \tag{16}$$

is estimated from routine closed-loop operating data. The residual variance from the model fitting provides an estimate of the minimum variance performance.

It may be necessary to include a constant in the regression equation to account for the fact that the average value of y_t may differ from the setpoint, y_{sp}. With either method, a number of performance indices can be defined. The methodology has been extended to account for regulation and setpoint tracking, feedforward variables, cascade systems, user-defined benchmarks, extended-horizon performance indices and multivariate systems. A comprehensive analysis of performance assessment methods and an extensive literature review is given in [1].

4. Extension to non-linear systems

4.1. Non-linear minimum variance control

The development of non-linear minimum variance controllers has been considered by a number of authors [9–11]. The latter two authors have considered systems that are the superposition of a non-linear process model plus a linear stochastic model of the form:

$$y_t = f_{P*}(u^*_{t-b}, w^*_t) + D_t \tag{17}$$

In this equation w^*_t denotes auxiliary variables that are known. The functional form of the plant model is quite general and can represent both linear and non-linear systems. The fundamental difference between $f_{P*}(u^*_{t-b}, w^*_t)$ and $f_P(y^*_{t-1}, u^*_{t-b})$ is that the latter model allows for non-linear dynamic terms that are functions of previous values of the process output. As noted by Grimble [11], $f_{P*}(u^*_{t-b}, w^*_t)$ does not need to be structured. It may be represented by non-linear difference equations, fuzzy neural networks (see also [10]), the output of a computer code or a simple look-up table (some restrictions will be noted in a subsequent section).

The derivation of the minimum variance controller for a process described by Eq. (17) is straightforward [11]. Eq. (17) can be written as:

$$y_{t+b} = f_{P*}(u^*_t, w^*_{t+b}) + D_{t+b}$$
$$= f_{P*}(u^*_t, w^*_{t+b}) + \widehat{D}_{t+b/t} + e_{t+b/t} \tag{18}$$

If it is possible to find the control action at time t such that $f_{P*}(u^*_t, w^*_t) + \widehat{D}_{t+b/t} = 0$, then the resulting controller is the minimum variance controller. All of the equations in the previous section apply with $\frac{\omega(q^{-1})}{\delta(q^{-1})}$ being replaced everywhere by $f_{P*}(u^*_t, w^*_{t+b})$. In this derivation $q^{-1} f_{P*}(u^*_{t-b}, w^*_t) = f_{P*}(u^*_{t-b-1}, w^*_{t-1})$.

It may not be possible to implement a minimum variance controller due to the fact that: (i) the minimum variance controller often gives high gain, wide bandwidth and unrealistically large control signal variations, and (ii) the controller is physically forbidden to take on certain values which are needed to achieve the minimum variance performance. In both of these cases, the variance of the output must exceed that for when it is possible to implement the minimum variance controller. However, σ^2_{MV}, Eq. (12), provides a theoretical lower bound on the output variance and can be used as a useful guide for controller assessment.

Pragmatic reasons are used to justify the linear representation for the disturbance [11]: (i) the disturbance model is a linear, time invariant approximation to the disturbances, (ii) it may not be possible to obtain a more sophisticated model given the quality of the data, and (iii) the resulting controllers appear to perform well when this approximation has been used. In addition to these guidelines, a number of theoretical considerations can be used to justify this choice of disturbance representation. These are discussed more fully in the following section.

Minimum variance design for the case where the process plus disturbance model has the more general form described in Eq. (3) has been considered in [9]. (These authors consider a self-tuning or adaptive form of the controller.) The methodology requires construction of the b-step ahead prediction and then finding the control action to set this to a specified value. Although conceptually similar to the method outlined in [10,11] additional complication arise due to the more general disturbance model.

4.2. Minimum variance performance bounds

The utility of the minimum variance performance estimate is that this lower bound on performance can be estimated readily from a representative sample of closed-loop data when it is felt that the process is adequately described by a linear transfer function with additive stochastic disturbance. In these instances, calculation of the bound requires that existence of a feedback invariant and a methodology to provide for its estimation from routine operating data. In this section, the results in [6] are extended to a class of non-linear systems. In order for a feedback invariant to exist it is necessary that the b-step ahead prediction error be independent of the manipulated variable action. As a first step, it is necessary to determine the class of systems for which this is true.

Theorem 1. *The b-step ahead prediction error, $b \geqslant 1$, is independent of the manipulated variable if the process plus disturbance admits a representation of the form*:

$$\delta(q^{-1})y_t = f(y^*_{t-b}, u^*_{t-b}) + \widetilde{D}_t \tag{19}$$

where $\widetilde{D}_t$ denotes a time series model of the form:

$$\phi(q^{-1})\nabla^d \widetilde{D}_t = a_t + \sum_{i=1}^{m} \theta_i a_{t-i} + \sum_{i=1}^{m}\sum_{j=i}^{m} \theta_{ij} a_{t-i} a_{t-j} + \cdots + \sum_{i_1=1}^{m} \cdots \sum_{i_k=i_{k-1}}^{m} \theta_{i_1 \ldots} a_{t-i_1} \cdots a_{t-i_k} \tag{20}$$

and $\{a_t\}$ is an white noise sequence with mean μ_a and variance σ_a^2. $\phi(q^{-1})$ and $\delta(q^{-1})$ are monic and stable polynomials in the backshift operator. The disturbance model must be invertible [18]. Stability and invertibility of non-linear polynomial ARMA models are discussed in [18,19].

Proof. See Appendix A □

Corollary 1-1. *The system described in Eq. (19) admits a description of the form*:

$$y_{t+b} = \hat{y}_{t+b/t} + e_{t+b/t} \tag{21}$$

where

$$\hat{y}_{t+b/t} = \delta^{-1}(q^{-1})(f(y^*_t, u^*_t)) + \widehat{D}_{t+b/t} \tag{22}$$

and

$$\delta(q^{-1})D_t = \widetilde{D}_t \tag{23}$$

The prediction error is given by:

$$e_{t+b/t} = y_{t+b} - \hat{y}_{t+b/t} = D_{t+b} - \widehat{D}_{t+b/t} \tag{24}$$

These follow immediately from Theorem 1 (Appendix). $\widetilde{D}_t$ can be interpreted as the additive disturbance and D_t as the output disturbance. (Note: It is not true that $\delta(q^{-1})\widehat{D}_{t+b/t} = \widehat{\widetilde{D}}_{t+b/t}$.)

Remarks

(1) By construction $E\{e_{t+b/t}\} = 0$ (Appendix)
(2) In the non-linear case $e_{t+b/t}$ in Eq. (24) is a very complex function. It will include terms of $a_{t+b}, \ldots, a_{t+1}$ and may include terms a_{t-k}, $k \geqslant 0$. The latter terms are not considered random variables when the conditional expectation is taken with respect to the information set I_t.
(3) In the paper there is also a requirement that the disturbance be modelled as a linear time series, or non-linear time series with a specific structure. The justification for this was considered by Grimble [11] who noted that pragmatically this is a reasonable assumption. This justification can be strengthened with the following comments.
 (a) The assumption of an additive term implies that the interaction terms between the disturbance terms and the dynamic terms are small in comparison to the main effect terms. As will be seen, this assumption can be tested empirically.
 (b) The assumption that the disturbance be modelled as the output of a linear or restricted complexity non-linear time series models is in fact not particularly restrictive. One of the most famous results in time series is a theorem due to Herman Wold, which states that a covariance stationary time series can always be represented as an infinite moving average process driven by white noise [16]. This theorem provides the foundations for the use of linear autoregressive moving average (ARMA) models. The Wold decomposition theorem applies even when the time series admits a non-linear representation since a non-linear time series always has a linear representation that will enable the second-order properties – autocorrelations – to be correctly modelled. Of course, it may be possible to produce smaller prediction errors using a non-linear time series. The consequence for this paper, is that if we use a linear time series for the disturbance, then the forecasts and forecast errors resulting from this representation will have the smallest forecast error, in the mean square error sense, among all linear forecasting methods. If we use a non-linear method for forecasts, then it may be possible to construct a lower prediction error.
(4) A sufficient condition for Theorem 1 is that the process description does not contain non-linear terms in y_{t+b-k}, $k = 1 \ldots (b-1)$, i.e. for any time periods over the delay horizon. Past dependencies on linear terms in y_{t+b-k}, $k \geqslant 1$ are permitted. Theorem 1 may not hold if the non-linear function $f(\cdot)$ in Eq. (19) includes non-linear terms in y_{t-b+i}, $i = 1 \ldots (b-1)$. This can be illustrated by the following simple example. Let

$$y_t = \phi y_{t-1} u_{t-2} + a_t, \quad 0 < \phi < 1 \tag{25}$$

If the distribution of $\{a_t\}$ is such that $a_t \geqslant 0$ and the input is restricted to positive values, then $y_t \geqslant 0\ \forall t$. With an obvious recursion:

$$\begin{aligned} y_{t+1} &= \phi y_t u_{t-1} + a_{t+1} \\ y_{t+2} &= \phi y_{t+1} u_t + a_{t+2} \\ &= \phi(\phi y_t u_{t-1} + a_{t+1}) u_t + a_{t+2} \end{aligned} \tag{26}$$

The corresponding conditional mean and prediction error are:

$$\begin{aligned} \hat{y}_{t+1/t} &= \phi y_t u_{t-1} + \mu_a \\ \hat{y}_{t+2/t} &= E\{y_{t+2}|I_t\} = \phi(\phi y_t u_{t-1} + \mu_a) u_t + \mu_a \\ e_{t+1} &= a_{t+1} - \mu_a \\ e_{t+2} &= a_{t+2} - \mu_a + \phi u_t (a_{t+1} - \mu_a) \end{aligned} \tag{27}$$

where μ_a is the mean of the driving force. The two-step ahead prediction error depends on the current control action. If y_{t-1} is changed to y_{t-2} in Eq. (25), then

$$\begin{aligned} \hat{y}_{t+1/t} &= \phi y_{t-1} u_{t-1} + \mu_a \\ \hat{y}_{t+2/t} &= \phi y_t u_t + \mu_a \\ e_{t+1} &= a_{t+1} - \mu_a \\ e_{t+2} &= a_{t+2} - \mu_a \end{aligned} \tag{28}$$

where μ_a is the mean of the driving force. The two-step ahead prediction error is independent on the current control action.

The dynamics in this example are members of the class of bilinear models [5]. Within this class dynamical models, superdiagonal and diagonal representations satisfy the conditions of Theorem 1.

Corollary 1-2. *For a linear ARIMA disturbance model* $\widetilde{D}_t = \frac{\theta(q^{-1})}{\phi(q^{-1})\nabla^d} a_t$, *the prediction error and conditional mean are given by*:

$$e_{t+b/t} = (1 + \varphi_1 q^{-1} + \cdots + \varphi_{b-1} q^{-(b+1)}) a_{t+b} \tag{29}$$

where the φ *weights are the impulse coefficients of the* $\frac{\theta(q^{-1})}{\phi(q^{-1})\nabla^d \delta(q^{-1})}$ *transfer function and*:

$$\begin{aligned} \delta(q^{-1})\hat{y}_{t+b/t} &= f(y_t^*, u_t^*) + \frac{P_b(q^{-1})}{\phi(q^{-1})\nabla^d} a_t \\ &= f(y_t^*, u_t^*) + \frac{P_b(q^{-1})}{\phi(q^{-1})\nabla^d} (y_t - \hat{y}_{t/t-1}) \\ &= f(y_t^*, u_t^*) + \frac{P_b(q^{-1})}{\theta(q^{-1})} \widetilde{D}_t \end{aligned} \tag{30}$$

$P_b(q^{-1})$ *is a polynomial in the backshift operator obtained by solving the Diophantine equation*:

$$\frac{\theta(q^{-1})}{\phi(q^{-1})\nabla^d \delta(q^{-1})} = 1 + \varphi_1 q^{-1} + \cdots + \varphi_{b-1} q^{-b+1} + \frac{q^{-b} P_b(q^{-1})}{\phi(q^{-1})\nabla^d \delta(q^{-1})} \tag{31}$$

These equations follow immediately from the definition of the conditional expectation and standard results for prediction of linear time series [7,8].

Theorem 2. *If the process in Eq.* (19) *is controlled by a linear or non-linear feedback controller* $g(\cdot)$, *then the prediction error* $e_{t+b/t}$ *is feedback invariant and can theoretically be recovered from routine operating data.*

Proof. The proof follows readily by noting that with a feedback controller $g(\cdot)$, the process between measured output and the disturbance is of the non-linear ARMA form:

$$\begin{aligned} y_{t+b} &= \delta^{-1}(q^{-1}) f(y_t^*, g(y_t^* - y_{\text{sp}})) + \widehat{D}_{t+b/t} + e_{t+b/t} \\ &= \hat{y}_{t+b/t} + e_{t+b/t} \qquad \square \end{aligned} \tag{32}$$

Corollary 2-1. *For a linear ARIMA disturbance* $\widetilde{D}_t = \frac{\theta(q^{-1})}{\phi(q^{-1})\nabla^d} a_t$, *the closed loop admits the following representations*:

Innovations or 1*-step ahead representation*

$$y_{t+1} = \hat{y}_{t+1/t} + a_{t+1} \tag{33}$$

where

$$\begin{aligned} \delta(q^{-1})\hat{y}_{t+1/t} &= f(y_{t-b+1}^*, g(y_{t-b+1}^* - y_{\text{sp}})) + \frac{P_1(q^{-1})}{\phi(q^{-1})\nabla^d} a_t \\ &= f(y_{t-b+1}^*, g(y_{t-b+1}^* - y_{\text{sp}})) \\ &\quad + \frac{P_1(q^{-1})}{\phi(q^{-1})\nabla^d} (y_t - \hat{y}_{t/t-1}) \end{aligned} \tag{34}$$

This equation is of the form:

$$A(q^{-1})\hat{y}_{t+1/t} = B(q^{-1}) f(y_{t-b+1}^*, g(y_{t-b+1}^* - y_{\text{sp}})) + C(q^{-1}) y_t \tag{35}$$

where $A(q^{-1}) = \theta(q^{-1})$, $B(q^{-1}) = \phi(q^{-1})\nabla^d$ *and* $C(q^{-1}) = P_1(q^{-1})$.

$$\delta(q^{-1})\hat{y}_{t+b/t} = f(y_t^*, g(y_t^* - y_{\text{sp}})) + \frac{P_b(q^{-1})}{\phi(q^{-1})\nabla^d} (y_t - \hat{y}_{t/t-1}) \tag{36}$$

b-step ahead representation

This equation is of the non-linear autoregressive (*NLAR*) *form*:

$$A(q^{-1})\hat{y}_{t+b/t} = B'(q^{-1}) f(y_t^*, g(y_t^* - y_{\text{sp}})) + C'(q^{-1}) y_t \tag{37}$$

where $B'(q^{-1}) = B(q^{-1})(1 + \varphi_1 q^{-1} + \cdots + \varphi_{b-1} q^{-(b+1)})$ *and* $C'(q^{-1}) = P_b(q^{-1})$.

Since $e_{t+b/t}$ is feedback invariant the minimum variance performance bounds can be estimated from routine operating data if it is possible to construct $\hat{y}_{t+b/t}$. There are significant challenges in doing this. The task is further complicated by noting that the $\hat{y}_{t+b/t}$ includes both exact values of output and deviation values. In linear systems, superposition applies, and it is easy to construct the predictor as the sum of disturbance and setpoint effects [20]. In the general non-linear case, the resulting model and model-building strategy may be quite difficult. Nonetheless, if the non-linear predictor can be approximated by a general function such as a universal polynomial approximation, it is possible and practicable that the minimum lower bounds can be estimated from the routine operating data. In the following section, the detailed procedures using Volterra series to estimate the minimum lower bounds are discussed.

5. Estimation of lower bound from operating data using Volterra series approximation

5.1. Volterra series approximation

A simpler case in which the non-linear function only includes the inputs is considered, i.e., $f(y^*_{t-b}, u^*_{t-b}) = f(u^*_{t-b})$. Volterra series are a very important class of functions which satisfy this condition. They are recognized as a powerful tool in the study of non-linear systems with memory and they have been used in a variety of situations both in applications and in the study of approximation of general non-linear systems [2,5,21–25].

Volterra series have theoretical justification as approximators with the following desirable properties:

(1) They can be used to model non-linear processes that have the following qualitative behavior [2,3]: (i) generate harmonics from periodic inputs, (ii) exhibit asymetric response to symmetric inputs, and (iii) possess input multiplicities.
(2) Many block-oriented non-linearities, such as Hammerstein (a static non-linearity, followed by a linear dynamic model), Weiner models (a linear dynamic model followed by a static non-linearity), Uryson models (Hammerstein models in parallel), and projection-pursuit models (Weiner models in parallel) have Volterra series representations. Recently, Vörös [26] has shown how Hammerstein systems can be modified to include systems with asymmetric dead-zones. These are very common non-linearities associated with valve-stiction and hysteresis.
(3) Parallel and cascade Volterra models result in Volterra models.
(4) Processes described by control-affine models, i.e. admit a Volterra representation. Non-linear control affine-models have been studied extensively in the control and chemical engineering literature, i.e. [27].
(5) Bilinear systems can be approximated with a structured Volterra representation [28]. However the full range of bilinear behavior cannot be modelled. A bilinear model exhibits input dependent stability, whereas a Volterra series does not show this behavior and a bilinear model, unlike a Volterra model, cannot have an input multiplicity [2,3].

Volterra series are being studied for applications in model reduction [29] and model-predictive control as they provided a relatively simple extension of linear systems to incorporate non-linearities. The design of a minimum variance controller for an industrial paper process using a Hammerstein model was examined in [30]. A number of investigations into using Volterra models for model predictive control have been reported: a polymerization system was considered in [21], a mineral processing systems [31], a simple heat exchanger systems [32], a ph process and high purity distillation [33,34]. In all cases, fundamental models were used to simulate the process and Volterra series models were used to build an approximate model that was used for (simulated) control purposes. Ref. [35] contains an extensive bibliography on recent results in non-linear identification for control purposes.

Consider the case when the process can be approximated as a finite discrete-time Volterra series with time delay b. The non-linear system in Eq. (19) can be written as:

$$\begin{aligned}\delta(q^{-1})y_t &= h_{m'}(u_t^*) + \widetilde{D}_t \\ h_{m'}(u_t^*) &= h_0 + \sum_{i=b}^{m'} h_i u_{t-i} + \sum_{i=b}^{m'}\sum_{j=i}^{m'} h_{ij} u_{t-i}u_{t-j} \\ &\quad + \cdots + \sum_{i_1=b}^{m'} \cdots \sum_{i_k=i_{k-1}}^{m'} h_{i_1\cdots i_k} u_{t-i_1}\cdots u_{t-i_k}\end{aligned} \tag{38}$$

Let y_{sp} denote output setpoint and define the deviation signal $\tilde{y}_t = y_t - y_{\mathrm{sp}}$. If it were possible to approximate the process plus controller with a finite Volterra series in the deviation variable $\tilde{y}_t$ then:

$$\begin{aligned}\delta(q^{-1})(\tilde{y}_t) &= h_{m'}(g_c(\tilde{y}_t)) - \delta(q^{-1})y_{\mathrm{sp}} + \widetilde{D}_t \\ &= \tilde{h}_0 + \sum_{i=b}^{m} \tilde{h}_i\tilde{y}_{t-i} + \sum_{i=b}^{m}\sum_{j=i}^{m} \tilde{h}_{ij}\tilde{y}_{t-i}\tilde{y}_{t-j} \\ &\quad + \cdots + \sum_{i_1=b}^{m}\cdots\sum_{i_k=i_{k-1}}^{m} \tilde{h}_{i_1\cdots i_k}\tilde{y}_{t-i_1}\cdots\tilde{y}_{t-i_k} + \widetilde{D}_t\end{aligned} \tag{39}$$

In the case of a linear disturbance model, the innovations representation, Eq. (33) and the b-step ahead representation, Eq. (36) have the following structure:

Innovations or 1-step ahead representation

$$\begin{aligned}A(q^{-1})\hat{\tilde{y}}_{t+1/t} &= h_0' + \sum_{i=0}^{m} h_i'\tilde{y}_{t-i} + \sum_{i=b-1}^{m}\sum_{j=i}^{m} h_{ij}'\tilde{y}_{t-i}\tilde{y}_{t-j} \\ &\quad + \cdots + \sum_{i_1=b-1}^{m}\cdots\sum_{i_k=i_{k-1}}^{m} h'_{i_1\cdots i_k}\tilde{y}_{t-i_1}\cdots\tilde{y}_{t-i_k}\end{aligned} \tag{40}$$

where $\hat{\tilde{y}}_{t+1/t} = \hat{y}_{t+1/t} - y_{\mathrm{sp}}$. In this equation, the non-linear effects are confined to time intervals $t-k$, $k \geqslant b-1$ and the linear effects start at $t-1$. In deriving this equation the nomenclature has been to use generic terms for the orders and parameters.

b-step ahead prediction

Polynomial autoregressive (PAR) representation

$$\begin{aligned}A(q^{-1})\hat{\tilde{y}}_{t+b/t} &= h_0' + \sum_{i=0}^{m} h_i'\tilde{y}_{t-i} + \sum_{i=0}^{m}\sum_{j=i}^{m} h_{ij}'\tilde{y}_{t-i}\tilde{y}_{t-j} \\ &\quad + \cdots + \sum_{i_1=0}^{m}\cdots\sum_{i_k=i_{k-1}}^{m} h'_{i_1\cdots i_k}\tilde{y}_{t-i_1}\cdots\tilde{y}_{t-i_k}\end{aligned} \tag{41}$$

In this representation, it is assumed that the process plus controller can be adequately modelled by a finite-order Volterra series.

Polynomial autoregressive with exogenous inputs (PARX) representation

$$A(q^{-1})\hat{\tilde{y}}_{t+b/t} = h''_0 + \sum_{i=0}^{m} h''_i \tilde{u}_{t-i} + \sum_{i=0}^{m}\sum_{j=i}^{m} h''_{ij}\tilde{u}_{t-i}\tilde{u}_{t-j} + \cdots + \sum_{i_1=0}^{m}\cdots\sum_{i_k=i_{k-1}}^{m} h''_{i_1\cdots i_k}\tilde{u}_{t-i_1}\cdots\tilde{u}_{t-i_k} + C'(q^{-1})\tilde{y}_t \tag{42}$$

In this representation, it is only necessary to assume that the process can be represented by a finite-order Volterra series. In both of the b-step ahead representations, the overlapping effects of the linear and non-linear effects start at the same time period.

While both representations can be used to estimate the variance of the b-step ahead prediction error, the direct b-step ahead representation provides a computationally attractive method. The estimate of the minimum-variance performance is the residual variance after fitting the model. This approach is the appropriate non-linear generalization of the method outlined in [17].

5.2. Polynomial AR model identification using orthogonal least squares methods

Direct methods to estimate the minimum lower bounds are used in this paper using the PAR model in Eq. (41) or PARX model in (42). Several methods have been proposed for this purpose: Orthogonal Least Squares (OLS) methods [36] and Fast Orthogonal Search (FOS) methods [37,38]. Use of Artificial Neural Network (ANN) models to approximate the NARMAX models is discussed in [39,40]. In this paper, the OLS methods are used to determine the order and estimate the parameters of the PAR/PARX models which represent the b-step ahead prediction. When $A(q^{-1}) = 1$, the linear-in-parameters model in Eqs. (41) and (42) can be written matrix form as:

$$\tilde{Y} = F\Theta + \Xi \tag{43}$$

with

$$\tilde{Y} = \begin{bmatrix} \tilde{y}_n \\ \vdots \\ \tilde{y}_N \end{bmatrix}, \quad F = \begin{bmatrix} 1 & \tilde{y}_{n-1} & \dots & \tilde{y}_{n-m}\cdots\tilde{y}_{n-m} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \tilde{y}_{N-1} & \dots & \tilde{y}_{N-m}\cdots\tilde{y}_{N-m} \end{bmatrix}$$

$$\Xi = \begin{bmatrix} e_n \\ \vdots \\ e_N \end{bmatrix}, \quad \Theta = \begin{bmatrix} h'_0 & h'_1 & \dots & h'_{m\dots m} \end{bmatrix} \tag{44}$$

F is the matrix of regression variables, Θ is the parameter vector and Ξ is the vector of b-step ahead prediction errors. N, n and M are the data length, starting point for the regression and number of regressor variables respectively ($n \geqslant M$). The parameter vector that minimizes $\|\tilde{Y} - F\Theta\|^2$ ($\|\cdot\|$ is the Euclidean norm) is given by:

$$\hat{\Theta} = (F^{\mathrm{T}}F)^{-1}F^{\mathrm{T}}\tilde{Y} \tag{45}$$

The number of possible terms in Eq. (44) could be very large. If the number of regressors is n_r and the maximum polynomial degree is n_d, the number of parameters is:

$$n_p = \frac{(n_r + n_d)!}{n_r!n_d!} \tag{46}$$

For example, if $n_r = 8$ and $n_d = 4$, then $n_p = 495$. A substantial reduction in the number of model parameters can be achieved by an appropriate selection of the orthonormal functions. The OLS algorithm is applicable when there are a 'reasonable' number of regressors. In the case of the larger model orders and higher polynomial degrees, the Fast Orthogonal Search (FOS) [38] and Genetic Programming (GP) [41] methods are recommended. Variations on least squares algorithms have also been developed for computational efficiency when there are a large numbers of candidate regressors, such as Volterra series, radial basis functions, neural networks or a polynomial NARX models [35].

The OLS algorithm developed by Chen et al. [36] involves a Q-R decomposition of the regression matrix F of the form $F = WA$, where W is an $M \times M$ upper triangular matrix and A is an $(N - n + 1 \times M)$ matrix with orthogonal columns in the sense that $W^{\mathrm{T}}W = D$ is a diagonal matrix. ($N - n + 1$ is the length of Y vector and M is the number of regressors.) After this decomposition one can calculate the OLS auxiliary parameter vector $\hat{g}$ as:

$$\hat{g} = D^{-1}W^{\mathrm{T}}\tilde{Y} \tag{47}$$

where $\hat{g}_i$ is the corresponding element of the OLS solution vector. The sums of squares of the observed values, $\tilde{Y}^{\mathrm{T}}\tilde{Y}$ can be written as:

$$\tilde{Y}^{\mathrm{T}}\tilde{Y} = \sum_{i=1}^{M}\hat{g}_i^2 w_i^{\mathrm{T}}w_i + \hat{\Xi}^{\mathrm{T}}\hat{\Xi} \tag{48}$$

where $\hat{\Xi}$ is the residual error from the full model, i.e. $\hat{\Xi} = \tilde{Y} - F\hat{\Theta}$. An error reduction ratio, $[\mathrm{err}]_i$ of F_i term can be defined as:

$$[\mathrm{err}]_i = \frac{\hat{g}_i w_i^{\mathrm{T}} w}{\tilde{Y}^{\mathrm{T}}\tilde{Y}} \tag{49}$$

This ratio offers a simple mean of order and select the model terms of a linear in parameters model according to their contribution to the performance of the model. The terms which have very small error reduction values, say smaller than ρ, are eliminated. The value of ρ determines how many terms will be included in the final model. This OLS algorithm can be interpreted as a forward-selection method where the reduction in sums of squares is maximized at each decision stage.

Alternatively, one can combine the OLS approach with a technique that provides a penalty for increasing model complexity. One such method is Akaike's Information Criterion AIC(s) [36]:

$$\mathrm{AIC}(\lambda) = N \ln \hat{\sigma}^2_{\Xi} + M \cdot \lambda \tag{50}$$

where M is the number of the model parameters, and $\hat{\sigma}^2_{\Xi}$ is the residual error. λ is a positive value chosen to provide a penalty for model complexity. Using statistical arguments, a value of $\lambda = 4$ is recommended [36,42].

It has been assumed that $A(q^{-1}) = 1$. This is not restrictive. The effect of ignoring this term is to increase the number of terms in the Volterra series. When some roots of $A(q^{-1})$ are close to the unit circle, the OLS approach may provide an inefficient form for approximation, requiring a large number of terms for the predictor in Eqs. (40) and (41). The more comprehensive model polynomial ARMA/X models can be employed to efficiently fit the closed-loop non-linear system in Eqs. (38) and (39) efficiently. The detailed iteration procedures can be found in [36].

For the invariant bounds to exist, it is necessary that the prediction error not depend on the manipulated variable. This assumption can be tested by regressing the prediction errors $\widehat{\Xi} = \widetilde{Y} - F\widehat{\Theta}Y^{\mathrm{T}}$ on values of the manipulated variable, and analyzing the regression results for statistical significance.

6. Simulation results

In this section, an example is provided to demonstrate the methodology outlined in this paper. Consider a non-linear dynamic system which can be represented by a second order Volterra series as:

$$\begin{aligned} y_t = {} & 0.2u_{t-3} + 0.3u_{t-4} + u_{t-5} + 0.8u^2_{t-3} + 0.8u_{t-3}u_{t-4} \\ & - 0.7u^2_{t-4} - 0.5u^2_{t-5} - 0.5u_{t-3}u_{t-5} + \widetilde{D}_t \end{aligned} \tag{51}$$

The disturbance is an ARIMA(2, 0, 0) process:

$$\widetilde{D}_t = D_t = \frac{a_t}{1 - 1.6q^{-1} + 0.8q^{-2}} \tag{52}$$

a_t is a white noise sequence with zero mean and variance 0.1. The true value of the minimum variance lower bound is 0.6656. A proportional (P) controller and a proportional-integral (PI) controller are used to control the simulated process:

$$\begin{aligned} u_t &= -0.2(y_t - y_{\mathrm{sp}}) \\ u_t &= -\frac{0.3 - 0.2q^{-1}}{1 - q^{-1}}(y_t - y_{\mathrm{sp}}) \end{aligned} \tag{53}$$

Fig. 1 shows a realization of D_t, y_t and u_t for the case when the PI controller was used. The open loop step response to a change in the manipulated variable of ±0.5 is shown in Fig. 2a and the closed-loop response with the PI controller to step changes of ±0.5 in setpoint zero is shown in Fig. 2b. The closed-loop response with the PI controller to a ±0.2 impulse disturbance is shown in Fig. 3. We notice that although the open-loop process is clearly non-linear, the non-linearity in the closed-loop process has been reduced.

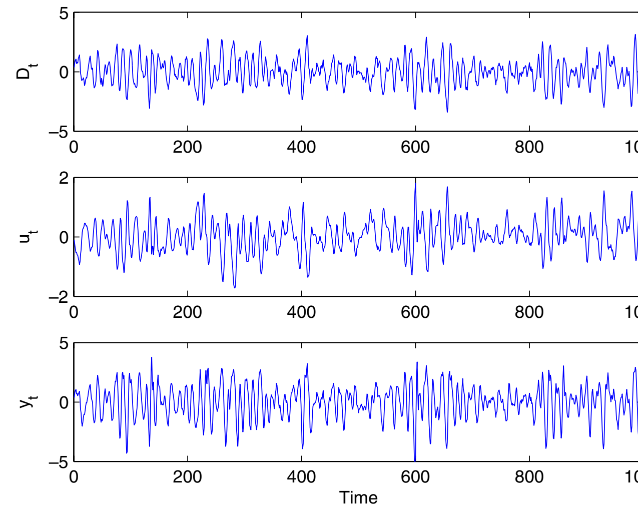

Fig. 1. The stochastic realizations of D_t, u_t and y_t with the PI controller.

For estimating the minimum variance lower bounds, three direct estimation methods are used:

- Linear autoregressive (LAR) model: $\hat{\tilde{y}}_{t+b/t} = \sum_{i=0}^{m} \gamma_i \tilde{y}_{t-i}$. For a linear model, it is also convenient to fit the data using an ARMA representation. Calculation of the variance of the b-step ahead forecast error is straightforward once the model parameters have been identified [6].
- PAR model (linear and quadratic terms only) (see Eq. (41)).
- PARX model (linear and quadratic terms only) (see Eq. (42)).

Five hundred observations were used to fit the parameters for these models. The minimum variance bound was calculated as the residual variance from each model. This procedure was repeated five hundred times with a different white-noise sequence. For the case when the PI controller was used, the estimates of the means of inputs and outputs are 3.15E−5 and 2.93E−4 respectively, and the estimates of the variance of inputs and outputs are 0.4233 and 3.2061.

When formulating the model, a large number of candidate terms are initially allowed, Table 1. In these examples, the AIC criterion was used in conjunction with OLS. For the PI controller case, the average number of the terms selected to fit the models are 14.5, 32.9 and 24.6 respective for the LAR, PAR and PARX models. For the P controller case the number of terms are 4.2, 10.8 and 5.6 for the LAR, PAR and PARX models respectively.

The estimates of the σ^2_{MV} using three models for the P and PI controller cases are shown in Table 2. The comparative box plots of the quality estimates are shown in Fig. 4. From the data in Table 2 and with reference to Fig. 4, the following observations can be made:

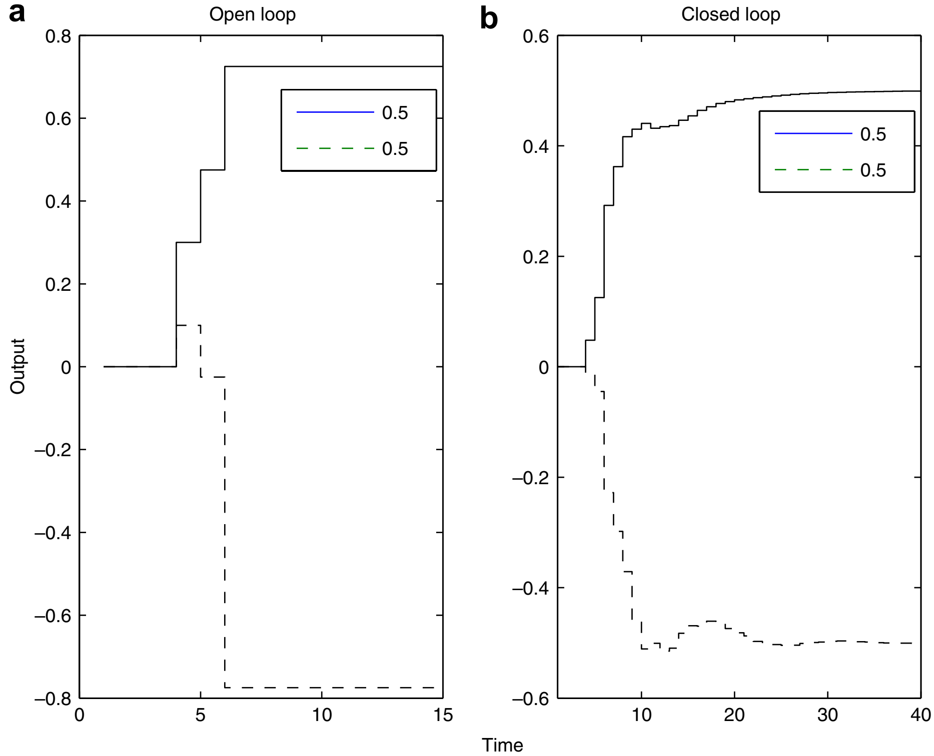

Fig. 2. Open loop and closed-loop responses (PI controller).

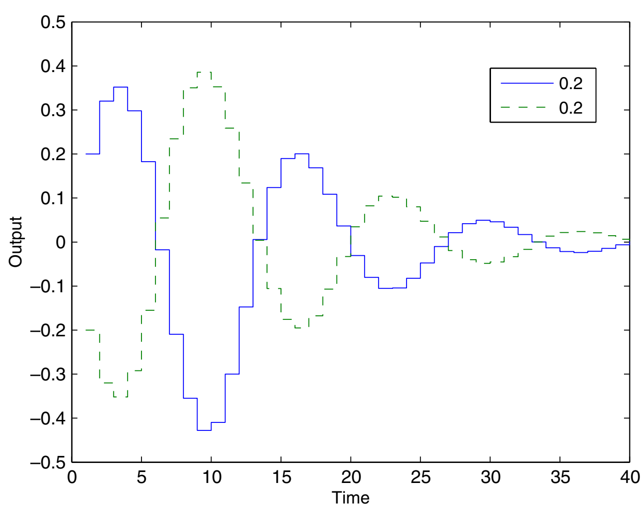

Fig. 3. Closed-loop simulation for a disturbance impulse change ±0.2 with the PI controller.

(1) When using linear AR models, the bias in the estimate of the minimum variance lower bounds in P controller case is much smaller than PI controller case. The reason is that the quadratic contributions are less important using the P controller than the PI controller. While it is possible that the inclusion of more autoregressive terms might provide a 'better' estimate σ^2_{MV} when using the linear models, the AIC criterion is used to provide a practical means of avoiding overfitting the data.
An ARMA($p,p-1$) model was fit to the data with increasing values of p. The model order, p, was chosen to minimize the AIC criterion. The variance of the b-step ahead forecast error was obtained from this model using standard techniques. The results were essentially indistinguishable from those obtained using the method described in the previous paragraph.

(2) There is a slight bias of the estimate of the performance bound for the PI controller case using PAR models. With the PI controller, the b-step prediction should include the infinite linear and quadratic autoregressive terms for our example. However, in our implementation, only finite terms were used.

Table 1
Initial candidate terms for models

	Linear (y)	Quadratic	Linear (u)	Total
LAR	40	0	0	40
PAR	20	55	0	75
PARX	20	55	20	95

Table 2
Estimates of σ^2_{MV} using different models

	PI controller		P controller	
	$\hat{\sigma}^2_{MV}$	s.d.	$\hat{\sigma}^2_{MV}$	s.d.
LAR	1.4487	0.4079	0.6826	0.0511
PAR	0.6964	0.1070	0.6697	0.0459
PARX	0.6704	0.0888	0.6683	0.0460

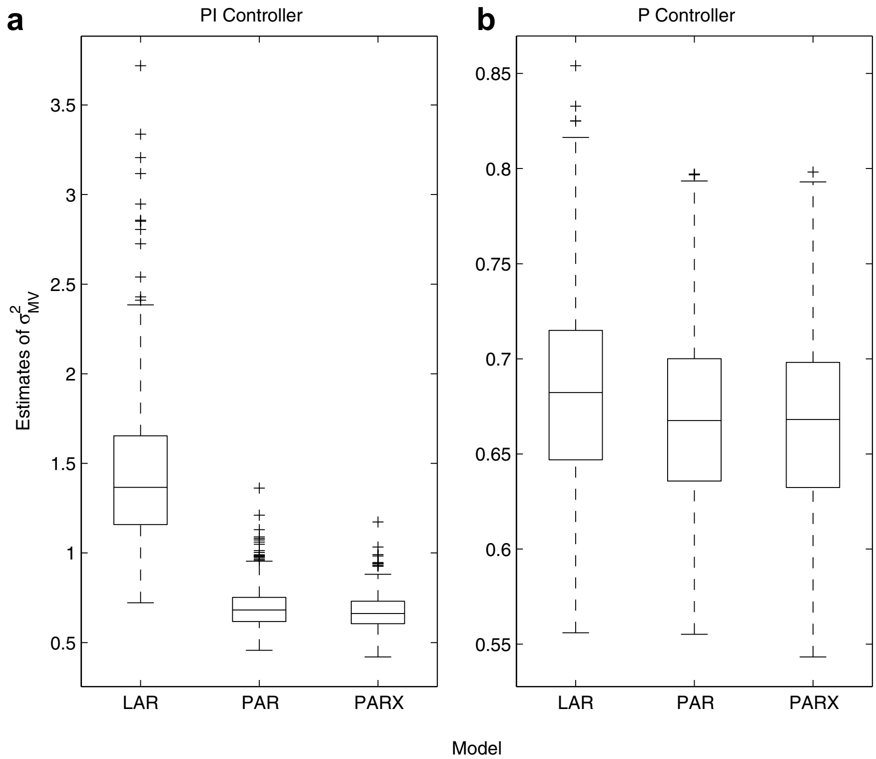

Fig. 4. Comparative box plots of the quality estimates from three models.

(3) The estimates using PARX models are better than PAR models specially for PI controller case since the PARX models are closer to the true prediction functions in finite linear and quadratic inputs form.

7. Discussion and conclusions

A class of non-linear dynamic/stochastic systems for which there exist minimum variance feedback invariant performance bounds has been established. For this class of systems, the minimum variance bound can be estimated using closed-loop data using a non-linear PAR or PARX model to estimate the *b*-step ahead prediction of the process. It is necessary to know the process delay. Application of the methodology to a simulation example indicates that this approach gives very credible estimates of the minimum variance performance bound. The simulation results indicate that the orthogonal-search method is effective. The theoretical developments also indicate that there are many challenges compared to the linear case. Specifically, issues related to structure verification and the inclusion of variable set-points in the analysis require attention.

How and where to use these results? The methodology can be used to quantify the effect that non-linearities in the closed loop have on the predictable component of the process. If the differences between the linear and non-linear performance bounds are important, then further investigation into the source of the non-linearities might be justified, or the use of a non-linear controller might be considered. In the case of linear systems, extended horizon performance indices have proven to be valuable [17] for two reasons: (i) the deadtime many not be exactly known and the extended horizon performance indices provide some guidance as to the sensitivity of the method to deadtime, and (ii) many model-based controllers, such as model predictive control used a prediction horizon as one of the tuning parameters. For these types of control algorithms, the extended-horizon performance bounds provide a performance evaluation method that is more closely aligned with the tuning objectives. In econometrics, it has been observed that non-linear models that provide good predictions over one time horizon, can provide poor predictions over extended horizons [43]. Further development of the methods outlined in the current paper to extended-horizon predictions would provide an explicit indication of the importance of non-linearities over extended time steps. Since one is interested in the impact of non-linearities on prediction effectiveness, it is not necessary that the dynamics or stochastic model satisfy the requirements of Theorem 1. To address the issues outlined in [43], it might be necessary to contemplate different model structures for each prediction horizon.

Although not investigated in this paper, the use of higher-order spectra [15,44,45] might prove valuable in deciding whether non-linear modelling is justified. An important assumption in this paper is that the disturbance appears as an additive, output-type disturbance. The use of cross-correlation techniques for testing for linear and non-linear minimum variance control [7,9] would appear to provide a constructive test for validating this assumption. There are many challenges in more broadly applying this method, including extension to set-point tracking and feedforward systems.

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Appendix A

Proof of Theorem 1. The expression for y_{t+b} is written down by inspection from Eq. (19). When $\delta(q^{-1})$ and $\phi(q^{-1})$ are stable, τ_j, $j \geqslant 0$ form a convergent series. It can be readily shown that by substituting for all values of y_{t+b-i}, $i = 1..(b-1)$:

$$y_{t+b} = \delta^{-1}(q^{-1})f(y_t^*, u_t^*) + \sum_{j=0}^{\infty} \tau_j(f_D(a^*_{t+b-1-j}) + a_{t+b-j}) \tag{A.1}$$

where τ_j is the jth impulse coefficient of $[\delta(q^{-1})\ \phi(q^{-1})\nabla^d]^{-1}$. In the more general case which admits the possibility that $\delta(q^{-1})$ is not stable or $d \geqslant 1$, the $\{\tau_j\}$ do not form a convergent series. In the general case:

$$\begin{aligned} y_{t+b} &= \delta^{-1}(q^{-1})f(y_t^*, u_t^*) \\ &\quad + \sum_{j=0}^{b-1} \tau_j(f_D(a^*_{t+b-1-j}) + a_{t+b-j}) + K_b(D_t, a_t^*) \end{aligned} \tag{A.2}$$

where $K_b(D_t, a_t^*)$ is a remainder term that is obtained by successive substitutions. (This representation is also valid for the case where $\{\tau_j\}$ form a convergent series.) The output disturbance has been represented as:

$$D_{t+b} = \sum_{j=0}^{b-1} \tau_j(f_D(a^*_{t+b-1-j}) + a_{t+b-j}) + K_b(D_t, a_t^*) \tag{A.3}$$

First, consider the 1-step conditional prediction of the disturbance D_{t+1} given the information set I_t.

$$\widehat{D}_{t+1/t} = \mu_a + K_1(D_t, a_t^*) \tag{A.4}$$

Since $\tau_0 = 1$ and $E\{K_1(D_t, a_t^*)|I_t\} = K_1(D_t, a_t^*)$. This latter expression results from the definition of the conditional expectation. The b-step ahead forecast is:

$$\begin{aligned} \widehat{D}_{t+b/t} &= E\left\{\sum_{j=0}^{b-1} \tau_j(f_D(a^*_{t+b-1-j}) + a_{t+b-j})|I_t\right\} + E\{K_b(D_t, a_t^*)|I_t\} \\ &= E\left\{\sum_{j=0}^{b-1} \tau_j(f_D(a^*_{t+b-1-j}) + a_{t+b-j})|I_t\right\} + K_b(D_t, a_t^*) \end{aligned} \tag{A.5}$$

Now in the above equation $E\{a_{t+k}|I_t\} = \mu_a$, $k = 1..b$ and $E\{a_{t-k}|I_t\} = a_{t-k} = D_{t-k} - \widehat{D}_{t-k/t-k-1}, k \geqslant 0$. The optimal predictor is not given by the term $K_b(D_t, a_t^*)$, as those terms that arise from the summation in Eq. (A.5) must be included. It is now necessary to evaluate terms of the form $E\{f_D(a^*_{t+k})|I_t\}, k = 1..b-1$. Each of these terms requires evaluation of the integral:

$$E\{f_D(a^*_{t+k})|I_t\} = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{k} f_D(a^*_{t+k})p(a_{t+k},,,a_{t+1})da_{t+k}\cdots da_{t+1} \tag{A.6}$$

where $p(a_{t+k},,,a_{t+1})$ is the joint distribution of $a_{t+k}\cdots a_{t+1}$.

Combining these results with Eq. (A.1):

$$\hat{y}_{t+b/t} = \delta^{-1}(q^{-1})(f(y_t^*, u_t^*)) + \widehat{D}_{t+b/t} \tag{A.7}$$

and

$$\begin{aligned} e_{t+b/t} &= y_{t+b} - \hat{y}_{t+b/t} \\ &= \sum_{j=0}^{b-1} \tau_j(f_D(a^*_{t+b-1-j}) - E\{f_D(a^*_{t+b-1-j})|I_t\} + a_{t+b-j} - \mu_a) \end{aligned} \tag{A.8}$$

The prediction error is feedback invariant thus proving the theorem. We also notice from Eq. (A.8) that $E\{e_{t+b/t}\} = 0$.

To illustrate the methodology, consider an ARIMA(0,1,2) disturbance.

$$D_t = \frac{(1 + h_1q^{-1} + h_2q^{-2})}{1 - q^{-1}}a_t \tag{A.9}$$

For this example $\tau_j = 1, j \geqslant 0$. The disturbance 2-steps into the future can be written as:

$$\begin{aligned} D_{t+2} &= \sum_{j=0}^{1} \tau_j(a_{t+2-j} + h_1a_{t+1-j} + h_2a_{t-j}) + D_t \\ &= a_{t+2} + h_1a_{t+1} + h_2a_t + a_{t+1} + h_1a_t + h_2a_{t-1} + D_t \end{aligned} \tag{A.10}$$

Using the definition of the condition expectation, and assuming that $\{a_t\}$ are a zero mean white noise sequence:

$$\begin{aligned} \widehat{D}_{t+2/t} &= h_2a_t + h_1a_t + h_2a_{t-1} + D_t \\ &= (h_1 + h_2)a_t + h_2a_{t-1} + D_t \end{aligned} \tag{A.11}$$

The prediction error is

$$e_{t+2/t} = a_{t+2} + h_1a_{t+1} + a_{t+1} = a_{t+2} + (1 + h_1)a_{t+1} \tag{A.12}$$

Equivalent expressions are obtained using the Diophantine equation approach.

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